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Generalised G_2-structures and type IIB superstrings

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Abstract: The recent mathematical literature introduces generalised geometries which are defined by a reduction from the structure group SO(d,d) of the vector bundle $T^d \oplus T^{d*}$ to a special subgroup. In this article we show that compactification of IIB superstring vacua on 7-manifolds with two covariantly constant spinors leads to a generalised G_2-structure associated with a reduction from SO$(7,7)$ to $G_2 \times G_2$. We also consider compactifications on 6-manifolds where analogously we obtain a generalised SU(3)-structure associated with SU(3) $\times$ SU(3), and show how these relate to generalised G_2-structures.

Keywords: Superstrings and Heterotic Strings, Superstring Vacua, Supersymmetry and Duality, Differential and Algebraic Geometry.

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1. Introduction

From a duality and a phenomenological point of view, the idea of compactifying superstring theories and M-theory is a rather appealing one. It also points to interesting geometrical issues as requiring a certain amount of supersymmetry to be preserved puts constraints on the internal background geometry and thus leads to special G-structures. For instance, compactification on a 7- or 6-manifold together with $\mathcal{N} = 1$ supersymmetry[1] yields a dilaton and a Killing spinor equation which induces a G_2- or SU(3)-structure with various non-trivial torsion classes.

For $\mathcal{N} = 2$ supersymmetry we basically need two special G-structures inside a given metric structure [6]. This naturally makes one consider the class of so-called *generalised geometries*, a concept which goes back to [13]. Over a d-manifold M^d, contraction on the vector bundle $T^d \oplus T^{d*}$ defines an inner product of signature (d, d) and therefore induces an $\mathrm{SO}(d, d)$-structure over M^d. A generalised geometry is then described by the (topological) reduction to a special subgroup of $\mathrm{SO}(d, d)$ together with a suitable integrability condition. In many situations, this can then be rephrased by the existence of two G-structures with prescribed torsion classes and which sit inside a given principal $\mathrm{SO}(n)$- or $\mathrm{Spin}(n)$-fibre bundle (cf. [8, 19]).

[1] Here, $\mathcal{N}$ denotes the number of covariantly constant spinors in the internal space.

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In the present article we investigate type IIB superstring vacua by compactifying on seven and six dimensional manifolds.[2] We shall focus on the geometrical structure of the vacuum space admitting a certain amount of supersymmetry in the external space, which can be achieved by a spinorial formulation of the supersymmetry variations. An additional analysis of the equations of motion single out physical vacua.

As we will explain, considering the "doubled" vector bundle $T \oplus T^*$ accounts for $\mathcal{N}=2$ supersymmetry by reducing the structure group from $\mathrm{SO}(7,7)$ or $\mathrm{SO}(6,6)$ to $G_2 \times G_2$ or $\mathrm{SU}(3) \times \mathrm{SU}(3)$ in the same way as compactification with $\mathcal{N}=1$ supersymmetry is accounted for by a reduction from the structure group $\mathrm{SO}(7)$ or $\mathrm{SO}(6)$ to G_2 or $\mathrm{SU}(3)$. Our starting point are the papers [10]–[12], where Hassan investigated T-duality issues along d directions. This naturally leads to the consideration of "general" supersymmetry transformations invariant under the action of the so-called *generalised T-duality* or *Narain group* $\mathrm{SO}(d,d)$. Here, "generalised" means that we treat the left- and right-moving sectors of the worldsheet independently under the supersymmetry variations. This is similar in spirit to the investigation of topologically twisted non-linear sigma models on target spaces admitting a generalised complex structure in the sense of Hitchin [14] and Gualtieri [8], see for instance [1, 4, 15, 16].

By taking Hassan's supersymmetry variations, we see that preserving supersymmetry on the external space requires the variations of the gravitinos $\Psi_{\pm X}$ and dilatinos $\lambda_\pm$ to vanish, i.e.

$$\delta_\pm \Psi_{\pm X} = 0 \qquad \text{and} \qquad \delta_\pm \lambda_\pm = 0\,.$$

These equations were also derived by Gauntlett et al. [5] from the perspective of wrapped NS5-branes in IIB supergravity. There, the authors found a solution by assuming $\mathcal{N}=2$ supersymmetry in dimension $d=3$ from a classical $\mathrm{SU}(3)$-structure point of view.

In this article we shall then proceed as follows. In section 2 we introduce the generalised supersymmetry variations of type IIB following [10]–[12]. Neglecting the action of the RR-sector we use standard compactification methods in section 3 to obtain the model $\mathbb{R}^{1,2} \times M^7$ together with the equations

$$\left(\gamma^a \partial_a \phi \mp \frac{1}{12} \gamma^{abc} H_{abc}\right) \eta_\pm = 0\,, \qquad \left(\partial_a + \frac{1}{4}\left(\omega_{abc} \mp \frac{1}{2} H_{abc}\right) \gamma^{bc}\right) \eta_\pm = 0 \tag{1.1}$$

for two spinors $\eta_\pm$ on the internal background. In section 4 we will discuss the notion of a generalised G_2-structure [19] and give an equivalent formulation of equations (1.1) by means of differential forms of mixed degree. As a quick illustration we shall apply this formulation to the setup taken from [5] in section 5. We then discuss compactifications on 6-manifolds which lead to generalised $\mathrm{SU}(3)$-structures in section 6. Issues about lifting string theory to generalised topological M-theory admitting a generalised G_2-structure (see for instance [3]) naturally rises the question about the relation between generalised G_2- and $\mathrm{SU}(3)$-structures which will be addressed in section 7.

The authors would like to acknowledge S.F. Hassan and R. Blumenhagen for useful discussions. Some ideas are based on the second author's doctoral thesis [20] who wishes to thank his supervisor Nigel Hitchin.

[2] For previous work see for instance [7, 9, 17].

2. Preliminaries

We briefly set up the notation for the type IIB theory following Hassan [10] (see also [11, 12]). Let us consider the gravitinos $\Psi_\pm$, the dilatinos $\lambda_\pm$ and the supersymmetry parameters $\epsilon_\pm$. They all arise as the real and imaginary part of the complex Weyl spinors

$$\Psi_X = \Psi_{+X} + i\Psi_{-X},\ X \in TM^{1,9}, \qquad \lambda = \lambda_+ + i\lambda_- \qquad \text{and} \qquad \epsilon = \epsilon_+ + i\epsilon_- .$$

The gravitinos (of spin 3/2) and the dilatinos (of spin 1/2) originate from the (R, NS)- and (NS, R)-sectors. We shall use an additional subscript $+/-$ to indicate if the R-spin representation is induced by the left or right moving sector. For instance, Ψ_{+X} comes from the (R, NS)-sector.

In the following we want to treat the left and right moving sector independently with respect to space-time supersymmetry and we therefore introduce two supersymmetry variations $\delta_\pm$. In particular, δ_+ and δ_- act only on the left and right moving sectors and interchange $R \leftrightarrow NS$. Both the supersymmetry parameters $\epsilon_\pm$ and the gravitinos $\Psi_\pm$ are supposed to be of positive chirality, i.e.

$$\Gamma^{11}\epsilon_\pm = \epsilon_\pm \qquad \Gamma^{11}\Psi_\pm = \Psi_\pm$$

while

$$\Gamma^{11}\lambda_\pm = -\lambda_\pm ,$$

that is $\lambda_\pm$ is of negative chirality. We shall neglect the action of the RR-fields and consider only the closed NS-NS 3-form flux H, i.e. $dH = 0$, and the dilaton Φ. Therefore the supersymmetry variations of the (R, NS)- and (NS, R)-sector are given by

$$\begin{aligned} \delta_\pm \Psi_{\pm X} &= \nabla_X \epsilon_\pm \mp \frac{1}{4} X \lrcorner H \cdot \epsilon_\pm,\ X \in TM^{1,9} \\ \delta_\pm \lambda_\pm &= \frac{1}{2}\left(d\phi \mp \frac{1}{2} H\right) \cdot \epsilon_\pm . \end{aligned} \tag{2.1}$$

The vanishing of the supersymmetry variations, that is

$$\delta_\pm \Psi_{\pm X} = 0 , \qquad \text{and} \qquad \delta_\pm \lambda_\pm = 0 , \tag{2.2}$$

is necessary to characterise the background manifold $M^{1,9}$ in the vacuum case. To find a solution to (2.2), we shall make specific assumptions which will occupy us next.

3. Compactification on M^7

In this section, we compactify the theory on a 7-manifold M^7, that is we consider the direct product model $\mathbb{R}^{1,2} \times M^7$ where H and ϕ take non-trivial values only over M^7. We want to determine the constraints on the underlying geometry of the internal space M^7 imposed by the vanishing of the supersymmetry variations (2.2).

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To that end, we decompose the supersymmetry parameters $\epsilon_\pm \in \Delta^\pm_{M^{1,9}}$ accordingly, that is

$$\epsilon_\pm = \sum_N \xi^N_\pm \otimes \eta^N_\pm \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where ξ and η live in the irreducible spin representation $\Delta_{\mathbb{R}^{1,2}}$ and $\mathbf{8}$ of Spin$(1,2)$ and Spin(7) respectively, and $N \leq \dim \mathbf{8} = 8$.

We fix the 10-dimensional space-time coordinates X^M (M=0,...,9) and assume the background fields to be independent of X^μ ($\mu = 0,1,2$). Coordinates on the internal space will be labeled by X^a for $a = 3,\ldots,9$. We use the convention

$$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}\, \mathbb{I}_{32\times 32}$$

with signature $(-,+,\ldots,+)$. We choose the explicit gamma matrix representation

$$\Gamma^M = \begin{cases} \gamma_\mu \otimes \mathbb{I}_{8\times 8} \otimes \sigma_2 & : \quad \mu = 0,\ldots,2 \\ \mathbb{I}_{2\times 2} \otimes \gamma_a \otimes \sigma_1 & : \quad a = 3,\ldots,9 \end{cases}$$

where the (8×8)-matrices γ_a are imaginary. The SO$(1,2)$ gamma matrices γ_μ and the Pauli matrices σ_i are given by

$$\gamma_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Furthermore, we note the relations

$$\prod_\mu \gamma_\mu = -\mathbb{I}_{2\times 2} \qquad \prod_a \gamma_a = -i\,\mathbb{I}_{8\times 8}\,.$$

The chirality operator Γ^{11} is therefore $\Gamma^{11} = \mathbb{I}_{2\times 2} \otimes \mathbb{I}_{8\times 8} \otimes \sigma_3$.

With these splittings at hand we want to carry out the supersymmetry variations (2.1). The external part of the dilatino variation vanishes trivially. For the internal part, we first note the useful identity

$$\Gamma^{M_1M_2M_3} H_{M_1M_2M_3} = (\mathbb{I}_{2\times 2} \otimes \gamma^{abc} \otimes \sigma_1)\, H_{abc}$$

by means of which we immediately obtain the dilatino variations

$$\delta_\pm \lambda_\pm = \frac{1}{2}\left[\mathbb{I}_{2\times 2}\xi^N_\pm \otimes \left(\gamma^a \partial_a \phi \mp \frac{1}{12}\gamma^{abc}\, H_{abc}\right) \eta^N_\pm \otimes \sigma_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right].$$

The condition $\delta_\pm \lambda_\pm = 0$ is then equivalent to

$$\left(\gamma^a \partial_a \phi \mp \frac{1}{12}\gamma^{abc}\, H_{abc}\right) \eta^N_\pm = 0\,. \tag{3.1}$$

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Next we focus on the variation of the gravitinos $\delta_\pm \Psi_{\pm M}$. The flatness of $\mathbb{R}^{1,2}$ implies

$$\nabla_\mu \xi_\pm^N = 0.$$

This solves the external part, and consequently we are left with

$$\delta_\pm \Psi_{\pm a} = \mathbb{I}_{2\times 2}\, \xi_\pm^N \otimes \left(\partial_a + \frac{1}{4}\left(\omega_{abc} \mp \frac{1}{2} H_{abc}\right)\gamma^{bc}\right)\eta_\pm^N \otimes \begin{pmatrix}1\\0\end{pmatrix}.$$

Imposing the condition $\delta_\pm \Psi_{\pm X} = 0$ finally yields

$$\left(\partial_a + \frac{1}{4}\left(\omega_{abc} \mp \frac{1}{2} H_{abc}\right)\gamma^{bc}\right)\eta_\pm^N = 0\,. \tag{3.2}$$

In this article we shall deal with the case $N = 1$, i.e. with exactly two internal spinors $\eta_\pm$. Hence a solution consists of the internal background data $(M^7, g, H, \eta_\pm, \phi)$ satisfying (3.1) and (3.2), where g is a metric, H a closed 3-form, $\eta_\pm$ two unit spinors in the associated irreducible spin representation **8** and ϕ a scalar function.

Note that the considerations above can be easily modified to tackle the case of non-chiral type IIA theory which results in similar geometric conditions.

4. Generalised G_2-structures

In this section we want to formulate the geometry of the internal space M^7 in the language of *generalised G_2-structures* which we are going to define next. For details and proofs of the facts below we refer to [19, 20] where these structures where introduced.

Contraction on the vector bundle $T^d \oplus T^{d*}$ over an arbitrary d-manifold defines a natural inner product of signature (d, d) and induces a spinnable SO(d, d)-structure. An element $X \oplus \xi \in T^d \oplus T^{d*}$ acts on a form $\tau \in \Lambda^*$ by

$$X \oplus \xi \bullet \tau = X \lrcorner\, \tau + \xi \wedge \tau\,.$$

As this squares to minus the identity,[3] we obtain an isomorphism between Cliff$(T^d \oplus T^{d*})$ and End(Λ^*). Moreover, the irreducible spin representations of Spin(d, d) can be realised as

$$S^\pm = \Lambda^{ev,od} T^{d*}\,.$$

In this way, an even or odd form ρ may be regarded as a *spinor* for $T^d \oplus T^{d*}$. In dimension 7, we call the pair (M^7, ρ) a *generalised G_2-manifold* if the stabiliser of the even or odd form ρ is conjugate to $G_2 \times G_2$ inside Spin$(7, 7)$. Since a 2-form b can be naturally identified with an element in the Lie algebra $\mathfrak{so}(d, d)$, it acts on $S^\pm$ by wedging with the exponential $e^b \bullet \tau = (1 + b + b^2/2 + \ldots) \wedge \tau$. Hence, if ρ defines a generalised G_2-manifold, so does the transformed spinor $e^b \wedge \rho$. This displays a crucial feature of generalised geometries, namely that these can be naturally transformed by the action of both diffeomorphisms and 2-forms (cf. also [14] and [8]).

[3] We follow the usual convention in mathematics where unit elements in the Clifford algebra square to -1.

The data of the previous section links into the generalised setup as follows. Consider a riemannian 7-manifold (M^7, g) together with two unit spinors η_+ and η_- living in the spinor bundle associated with a fixed spin structure over M^7 and the irreducible Spin(7)-representation $\mathbf{8}$. In terms of G-structures, it is a well-known fact that each spinor induces a reduction to a principal G_2-fibre bundle inside this spin structure. For sake of clarity, we denote the associated structure groups by $G_{2\pm}$. The group $G_{2+} \times G_{2-}$ then fixes the element $\eta_+ \otimes \eta_-$ inside the irreducible $\mathrm{Spin}(7) \times \mathrm{Spin}(7)$-representation space $\mathbf{8} \otimes \mathbf{8}$. On the other hand, these groups also act on $\Lambda^{ev,od}$ via the inclusion into Spin(7, 7). In order to compare these two actions we write

$$\mathrm{Cliff}\left(T^d, g\right) \widehat{\otimes} \, \mathrm{Cliff}\left(T^d, -g\right) = \mathrm{Cliff}\left(T^d \oplus T^{d*}\right),$$

where the isomorphism is given by extension of the map

$$X \widehat{\otimes} Y \mapsto (X \oplus -X \lrcorner\, g) \bullet (Y \oplus Y \lrcorner\, g),$$

and $\bullet$ now also denotes multiplication in $\mathrm{Cliff}(T^d \oplus T^{d*})$. Let $(\varphi \otimes \psi)^{\mathrm{ev,od}}$ represent the even or odd part of the form obtained through fierzing and let $\cdot$ denote Clifford multiplication in $\mathbf{8}$. One can show that

$$(X \cdot \varphi \otimes \psi)^{\mathrm{ev,od}} = X \widehat{\otimes} 1 \bullet (\varphi \otimes \psi)^{\mathrm{od,ev}} \tag{4.1}$$

$$(\varphi \otimes Y \cdot \psi)^{\mathrm{ev,od}} = \pm 1 \widehat{\otimes} Y \bullet (\varphi \otimes \psi)^{\mathrm{od,ev}} \tag{4.2}$$

for any φ, $\psi \in \mathbf{8}$. Hence the $G_{2+} \times G_{2-}$-invariant tensor product $\eta_+ \otimes \eta_-$ induces elements $\rho_0^{ev,od} \in S^{\pm}$ whose stabiliser inside Spin(7, 7) is conjugate to $G_2 \times G_2$.

A particular instance of this construction was carried out in [5] where the authors considered the case of two orthogonal spinors η_+ and η_-, that is, the structure group reduces to an honest $G_{2+} \cap G_{2-} = \mathrm{SU}(3)$. In our situation, the choice of the two spinors is perfectly general. To fully appreciate that point, define the spinor $\widetilde{\eta}_+ = \eta_- - q(\eta_-, \eta_+)\eta_+$ which is orthogonal to η_+. It induces a vector field determined by the relation $X \cdot \eta_+ = \widetilde{\eta}_+$. Outside the zero locus of X, the pair $(\eta_+, \widetilde{\eta}_+/\ \|X\|)$ induces an SU(3)-structure on M which breaks down precisely when η_+ and η_- are parallel, that is, $G_{2+} \cap G_{2-} = G_2$. This is also reflected in the following explicit description of $\rho_0^{\mathrm{ev,od}}$ in terms of the underlying $G_{2+} \cap G_{2-}$-invariants. The coefficients of the form $\eta_+ \otimes \eta_-$ can be computed by

$$g(\eta_+ \otimes \eta_-, e_I) = q(e_I \cdot \eta_+, \eta_-),$$

where q denotes a suitably scaled Spin(7)-invariant inner product on $\mathbf{8}$ and $e_I = e_{i_1} \wedge \cdots \wedge e_{i_p}$ is a multi-index of an orthonormal basis for g. Since Spin(7) acts transitively on the set of pairs of orthonormal spinors, we may choose an orthonormal basis in $\mathbf{8}$ such that

$$\eta_+ = (1, 0, 0, 0, 0, 0, 0, 0, 0)^{\mathrm{tr}}, \qquad \text{and} \qquad \eta_- = (\cos(a), \sin(a), 0, 0, 0, 0, 0, 0)^{\mathrm{tr}}.$$

If the spinors η_+ and η_- are linearly independent, their isotropy groups G_{2+} and G_{2-} intersect in SU(3) which fixes a 1-form $\alpha = e_7$, a symplectic form $\omega = e_{12} + e_{34} + e_{56}$ and two 3-forms $\psi_+ = e_{135} - e_{146} - e_{236} - e_{245}$ and $\psi_- = e_{136} + e_{145} + e_{235} - e_{246}$. These are the real and the imaginary part of the SU(3)-invariant holomorphic volume form [2]. We then find

$$\begin{aligned}\rho_0^{\mathrm{ev}} &= c + s\omega - c\left(\psi_- \wedge \alpha + \frac{\omega^2}{2}\right) + s\psi_+ \wedge \alpha - s\frac{\omega^3}{6} \\ \rho_0^{\mathrm{od}} &= s\alpha - c(\psi_+ + \omega \wedge \alpha) - s\psi_- - s\frac{\omega^2}{2} \wedge \alpha + c\ \mathrm{vol}_g\,,\end{aligned} \tag{4.3}$$

where c and s are shorthand for $\cos(a)$ and $\sin(a)$, and $a = \sphericalangle(\eta_+, \eta_-)$ describes the angle between the spinors η_+ and η_-. The underlying SU(3)-structure fluctuates with a and breaks down when $s = 0$, i.e. the spinors are parallel. Consequently, only the forms $s\alpha$, $s\omega$ etc. are globally defined over M^7 and it follows that in general the SO(7)-structure does not reduce to a global "static" SU(3)-structure. Moreover, at a point where $a = 0$, i.e. $\eta = \eta_+ = \eta_-$, we have

$$\begin{aligned}\rho^{\mathrm{ev}} &= \ \ 1 - \star\varphi \\ \rho^{\mathrm{od}} &= -\varphi + \mathrm{vol}\,,\end{aligned}$$

with φ denoting the invariant 3-form of the G_2-structure defined by η. This explicit description also reveals how to relate the $G_2 \times G_2$-invariant forms ρ_0^{ev} and ρ_0^{od}. For a p-form ξ^p, define $\sigma(\xi^p)$ to be 1 for $p \equiv 0, 3 \bmod 4$ and -1 for $p \equiv 1, 2 \bmod 4$. A direct computation shows that

$$\star\sigma(\eta_+ \otimes \eta_-)^{\mathrm{ev,od}} = (\eta_+ \otimes \eta_-)^{\mathrm{od,ev}}\,.$$

In general, a $G_2 \times G_2$-invariant spinor ρ in S^+ or S^- determines a metric g and a 2-form b for which it can be uniquely written (up to a sign) as

$$\rho^{\mathrm{ev,od}} = e^{-\phi} e^b \wedge (\eta_+ \otimes \eta_-)^{\mathrm{ev,od}} \in S^\pm\,.$$

Therefore, any generalised G_2-manifold can be equivalently characterised by the set of data $(M^7, g, b, \eta_\pm, \phi)$. We refer to the induced 2-form b as the *B-field* of the generalised G_2-structure. In order to relate ρ^{ev} with ρ^{od} we introduce the *generalised Hodge-* or *box operator* $\Box_{g,b} : \Lambda^{ev,od} \to \Lambda^{od,ev}$ defined by

$$\Box_{g,b}\rho^{\mathrm{ev,od}} = e^b \wedge \star\sigma\left(e^{-b} \wedge \rho^{\mathrm{ev,od}}\right).$$

One can then show that

$$\Box_{\rho^{\mathrm{ev,od}}}\rho^{\mathrm{ev,od}} = \rho^{\mathrm{od,ev}}\,.$$

For sake of concreteness, we usually assume the $G_2 \times G_2$-invariant spinor ρ to be even and write ρ_0 for the B-field free form $\eta_+ \otimes \eta_-$ given by (4.3). We shall also use the sloppier notation $\Box_\rho$ for $\Box_{g,b}$ if g and b are induced by ρ. Note that the

$$g: 28, \quad b: 21, \quad \eta_+: 7, \quad \eta_-: 7, \quad \phi: 1$$

degrees of freedom sum to $64 = \dim \Lambda^{ev,od}$, so that this data effectively parametrises the *open* orbit of a $G_2 \times G_2$-invariant form under the action of $\mathbb{R}_{>0} \times \mathrm{Spin}(7,7)$. Following the language in [13] such a spinor is called *stable.* Stability allows us to consider a certain variational principle introduced by Hitchin [13] which also gained some attraction with a view towards a topological M-theory [3] (see also [18]). The variation takes place over a d_H-cohomology class, where H is a closed 3-form and $d_H = d + H\wedge$. If ρ is d_H-closed, then it defines a critical point for this variational problem if and only if $d_H\Box_\rho\rho = 0$. Note that if the spinors η_+ and η_- are equal and $H = 0$, this equation reduces to the classical condition $d\star\varphi = 0$ [13]. To see how this relates to (3.1) and (3.2), recall that the twisted Dirac operator over $\mathbf{8}\otimes\mathbf{8}$ transforms into $d + d^\star$ under fierzing. A more general argument taking into account the action of the 3-from H can then be invoked to show that $(M^7, g, H, \eta_\pm, \phi)$ satisfies (3.1) and (3.2) if and only if the corresponding $G_2 \times G_2$-invariant spinor $\rho_0 = (\eta_+ \otimes \eta_-)^{ev}$ satisfies

$$d_He^{-\phi}\rho_0 = de^{-\phi}\rho_0 + H\wedge e^{-\phi}\rho_0 = 0\,, \quad d_H\Box_{\rho_0}e^{-\phi}\rho_0 = d\Box_{\rho_0}e^{-\phi}\rho_0 + H\wedge\Box_{\rho_0}e^{-\phi}\rho_0 = 0\,, \tag{4.4}$$

that is, $e^{-\phi}\rho_0$ defines a critical point. In particular, we see that as a integrability condition we need

$$e^{-\phi}\cos(a) = \text{const}\,.$$

If H is globally exact, i.e. $H = db$, (4.4) can be written in the more succinct form

$$de^{-\phi}(e^b\wedge\rho_0) = 0\,, \qquad de^{-\phi}\Box_{g,b}(e^b\wedge\rho_0) = 0\,.$$

5. Recovering the classical SU(3)-case

Equations (3.1) and (3.2) were first derived by Gauntlett et al. [5] from a quite different point of view. Starting with IIB supergravity they studied wrapped NS5-branes over calibrated submanifolds inside an internal 7-manifold with an SU(3)-structure. As an illustration of the previous section, we reconsider their setup which turns out to be described by a "static" generalised G_2-structure with $a \equiv \pi/2$ (that is, the structure group reduces to a fixed SU(3)), together with an additional closed 3-form, the NS-NS flux H.

Under this assumption the form ρ defining the generalised G_2-structure becomes

$$\rho_0 = \omega + \psi_+\wedge\alpha - \frac{\omega^3}{6}$$

with associated odd form

$$\Box_{\rho_0}\rho_0 = \alpha - \psi_- - \frac{\omega^2}{2}\wedge\alpha\,.$$

The supersymmetry equations are equivalent to

$$d_He^{-\phi}\rho_0 = 0 \qquad \text{and} \qquad d_He^{-\phi}\Box_{\rho_0}\rho_0 = 0$$

which written in homogeneous components can then be rephrased by

$$\begin{aligned} d\omega &= d\phi\wedge\omega\,,\\ \psi_+\wedge d\alpha &= -d\phi\wedge\psi_+\wedge\alpha + d\psi_+\wedge\alpha - H\wedge\omega\,,\\ \frac{1}{2}d\omega\wedge\omega^2 &= d\phi\wedge\frac{\omega^3}{6} - H\wedge\psi_+\wedge\alpha \end{aligned}$$

and

$$
\begin{aligned}
d\alpha &= d\phi \wedge \alpha\,, \\
d\psi_- &= d\phi \wedge \psi_- + H \wedge \alpha\,, \\
d\omega \wedge \omega \wedge \alpha &= d\phi \wedge \alpha \wedge \frac{\omega^2}{2} - H \wedge \psi_-\,.
\end{aligned}
$$

We finally conclude

$$
\begin{aligned}
d(e^{-\phi}\alpha) &= 0\,, & d(e^{-\phi}\omega) &= 0\,, \\
\alpha \wedge d\psi_+ &= H \wedge \omega\,, & \frac{1}{3} d\omega \wedge \omega &= \alpha \wedge H \wedge \psi_+\,, \\
d(e^{-\phi}\psi_-) &= H \wedge \alpha\,, & d\omega \wedge \omega \wedge \alpha &= \psi_- \wedge H\,.
\end{aligned} \tag{5.1}
$$

The equations of motion are solved since H is closed, i.e. $dH = 0$, as proved in [5]. Therefore (5.1) characterises the physical vacua.

6. Compactification on M^6 and generalised $\mathrm{SU}(3)$-structures

Following the procedure of section 3 we can also compactify on a 6-dimensional manifold M^6. Recall that we have $\mathrm{Spin}(6) = \mathrm{SU}(4)$ and that the irreducible spin representations of positive and negative chirality $\Delta_\pm$ are just the $\mathrm{SU}(4)$-vector representation $\mathbf{4}$ and its conjugate $\overline{\mathbf{4}}$. The supersymmetry equations compactified on M^6 thus become

$$
\nabla^{LC}_X \eta_\pm \pm \frac{1}{4} X \lrcorner H \cdot \eta_\pm = 0\,, \qquad \left(d\phi \pm \frac{1}{2} H\right) \cdot \eta_\pm = 0 \tag{6.1}
$$

for two complex spinors $\eta_\pm$. Since we work in type IIB theory both η_+ and η_- are assumed to be of positive chirality. Similarly, we can consider type IIA theory by choosing the spinors to be non-chiral.

Recall that $\mathrm{SU}(4)/\mathrm{SU}(3) = S^7$, hence the choice of two unit spinors $\eta_\pm \in \mathbf{4}$ induces a reduction to two $\mathrm{SU}(3)_\pm$-subbundles. The $\mathrm{SU}(4)$-representations $\Delta_\pm$ decompose into $\mathbf{3}_\pm \oplus \mathbf{1}_\pm$ and $\overline{\mathbf{3}}_\pm \oplus \overline{\mathbf{1}}_\pm$. Consequently, we can also consider the corresponding $\mathrm{SU}(3)_\pm$-invariant spinors $\overline{\eta}_\pm \in \overline{\mathbf{4}}$. We want to describe the data $(M^6, g, H, \eta_\pm, \phi)$ in the language of generalised geometry where it gives rise to a generalised $\mathrm{SU}(3)$-structure. This is completely analogous to section 4 and the proofs of [19] carry over without difficulty. Again we content ourselves with a brief outline of the corresponding results.

Rather than working with the complex spinors we will consider the real $\mathrm{SU}(4)$-module S obtained by forgetting the complex structure on $\mathbf{4}$ or $\overline{\mathbf{4}}$, that is the complexification of S is just $S^{\mathbb{C}} = \mathbf{4} \oplus \overline{\mathbf{4}}$. Note that the riemannian volume element vol_g induces a complex structure on S and acts on $\mathbf{4}$ and $\overline{\mathbf{4}}$ by multiplication with i and $-i$ respectively. We let

$$
\varphi_\pm = \mathrm{Re}(\eta_\pm)\,, \qquad \widehat{\varphi}_\pm = \mathrm{Im}(\eta_\pm)\,,
$$

so that

$$
\mathrm{vol}_g \cdot \varphi_\pm = -\widehat{\varphi}_\pm\,.
$$

Since S carries an SU(4)-invariant riemannian inner product, we can identify $S \otimes S$ with $\Lambda^* T^{6*}$ through fierzing so that (4.2) holds. This yields two forms $\varphi_+ \otimes \varphi_-$ and $\varphi_+ \otimes \widehat{\varphi}_-$ which we can interpret as $\mathrm{SU}(3)\times\mathrm{SU}(3)$-invariant spinors and which we want to decompose into an even and an odd part. Note that under complexification of this isomorphism, the components $\mathbf{4}\otimes\mathbf{4}$ and $\overline{\mathbf{4}}\otimes\overline{\mathbf{4}}$ get mapped onto odd complex forms, while the off-diagonal components $\overline{\mathbf{4}}\otimes\mathbf{4}$ and $\mathbf{4}\otimes\overline{\mathbf{4}}$ become even since $\mathbf{4}$ and $\overline{\mathbf{4}}$ are dual to each other. Writing $\varphi_+ \otimes \varphi_- = (\eta_+ + \overline{\eta}_+) \otimes (\eta_- + \overline{\eta}_-)/4$ etc. we obtain

$$
\begin{aligned}
\tau_0 &= (\varphi_+ \otimes \varphi_-)^{\mathrm{ev}} = -(\widehat{\varphi}_+ \otimes \widehat{\varphi}_-)^{\mathrm{ev}} = \ \ \frac{1}{2}\,\mathrm{Re}(\eta_+ \otimes \overline{\eta}_-) \\
\widehat{\tau}_0 &= (\widehat{\varphi}_+ \otimes \varphi_-)^{\mathrm{ev}} = -(\varphi_+ \otimes \widehat{\varphi}_-)^{\mathrm{ev}} = \ \ \frac{1}{2}\,\mathrm{Im}(\eta_+ \otimes \overline{\eta}_-) \\
\upsilon_0 &= (\widehat{\varphi}_+ \otimes \widehat{\varphi}_-)^{\mathrm{od}} = -(\varphi_+ \otimes \varphi_-)^{\mathrm{od}} = -\frac{1}{2}\,\mathrm{Re}(\eta_+ \otimes \eta_-) \\
\widehat{\upsilon}_0 &= (\varphi_+ \otimes \widehat{\varphi}_-)^{\mathrm{od}} = \ \ (\widehat{\varphi}_+ \otimes \varphi_-)^{\mathrm{od}} = \ \ \frac{1}{2}\,\mathrm{Im}(\eta_+ \otimes \eta_-)\,.
\end{aligned}
$$

To see how these forms relate to each other, we note that in dimension 6 the $\Box_{g,b}$-operator respects the parity of the forms and satisfies $\Box_{g,b}^2 = -Id$, that is $\Box_{g,b}$ induces a complex structure on $\Lambda^* T^*$ (it is effectively the $\wedge$-operator introduced in [14]). We then have

$$
\Box_{g,b}\tau_b = \widehat{\tau}_b\,, \qquad \Box_{g,b}\upsilon_b = \widehat{\upsilon}_b\,,
$$

where $\tau_b = e^b \wedge \tau_0$ etc..

As in section 4 we can compute a normal form description which we can express in terms of the underlying $\mathrm{SU}(2) = \mathrm{SU}(3)_+ \cap \mathrm{SU}(3)_-$-invariants if the unit spinors $\eta_\pm$ are linearly independent. Using again the complexified isomorphism $S^{\mathbb{C}} \otimes S^{\mathbb{C}} \cong \Lambda^* T^{6*\mathbb{C}}$ and decomposing $\eta_- = c_1\eta_+ + c_2\eta_+^\perp$ with two complex scalars c_1, $c_2 \in \mathbb{C}$ we find

$$
\eta_+ \otimes \eta_- = i\bar{Z} \wedge \left(c_1\Omega + c_2 e^{i\omega_1}\right) \tag{6.2}
$$

and

$$
\eta_+ \otimes \overline{\eta}_- = e^{i\alpha\wedge\beta} \wedge \left(\bar{c}_1 e^{i\omega_1} + \bar{c}_2\Omega\right), \tag{6.3}
$$

where expressed in a suitable local orthonormal basis $e_1, \ldots, e_6$ we have the two real 1-forms $\alpha = e_5$, $\beta = e_6$, the complex 1-form $Z = e_5 + ie_6$, the self-dual 2-forms $\omega_1 = e_{12} + e_{34}$, $\omega_2 = e_{13} - e_{24}$, $\omega_3 = e_{14} + e_{23}$ and the complex symplectic form $\Omega = \omega_2 - i\omega_3$. The normal forms of $\overline{\eta}_+ \otimes \eta_-$ and $\overline{\eta}_+ \otimes \overline{\eta}_-$ are obtained by complex conjugation in $\Lambda^* T^{6*\mathbb{C}}$.

Finally we wish to state the supersymmetry equations (6.1) in terms of the $\mathrm{SU}(3) \times \mathrm{SU}(3)$-invariant forms τ_0, $\widehat{\tau}_0$, υ and $\widehat{\upsilon}$. The real version of (6.1) is given by

$$
\nabla^{LC}_X \varphi_\pm \pm \frac{1}{4} X \lrcorner H \cdot \varphi_\pm = 0\,, \qquad \left(d\phi \pm \frac{1}{2}H\right) \cdot \varphi_\pm = 0
$$

and

$$
\nabla^{LC}_X \widehat{\varphi}_\pm \pm \frac{1}{4} X \lrcorner H \cdot \widehat{\varphi}_\pm = 0\,, \qquad \left(d\phi \pm \frac{1}{2}H\right) \cdot \widehat{\varphi}_\pm = 0\,.
$$

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The same computation as in the generalised G_2-case shows that this is equivalent to

$$d_H e^{-\phi}\tau_0 = d_H e^{-\phi}\widehat{\tau}_0 = 0\,, \qquad d_H e^{-\phi}\upsilon_0 = d_H e^{-\phi}\widehat{\upsilon}_0 = 0\,,$$

that is

$$d_H e^{-\phi}\eta_+ \otimes \eta_- = 0\,, \qquad d_H e^{-\phi}\overline{\eta}_+ \otimes \eta_- = 0\,.$$

If H is globally exact, that is $H = db$, we can write these equations more succinctly as

$$de^{-\phi}\tau_b = de^{-\phi}\Box_{g,b}\tau_b = 0\,, \qquad de^{-\phi}\upsilon_b = de^{-\phi}\Box_{g,b}\upsilon_b = 0\,.$$

7. Dimension 6 vs. 7

The inclusion $\mathrm{SU}(3) \subset G_2$ allows one to pass from an SU(3)-structure in dim $= 6$ to a G_2-structure in dim $= 7$. In the same vein, the inclusion $\mathrm{SU}(3)\times\mathrm{SU}(3) \subset G_2 \times G_2$ relates generalised SU(3)- to generalised G_2-structures. In this final section we want to render this link explicit in both the spinorial and the form picture of a generalised structure. We first discuss the algebraic setup before we turn to integrability issues.

To start with, assume that we are given a generalised G_2-structure $(T, g, b, \eta_\pm, \phi)$ over the 7-dimensional vector space $T^7 = T$ together with a preferred unit vector α. We want to induce a generalised SU(3)-structure on $T^6 = \widetilde{T}$ defined by $T = \widetilde{T} \oplus \mathbb{R}\alpha$. Since $\alpha\cdot\alpha = -1$, the choice of such a vector induces a complex structure on the irreducible Spin(7)-module $\mathbf{8}$ which is compatible with the spin-invariant riemannian inner product. Hence the complexification of $\mathbf{8}$ is

$$\mathbf{8}\otimes\mathbb{C} = \Delta^{1,0} \oplus \Delta^{0,1}\,,$$

where

$$\Delta^{1,0/0,1} = \{\eta \mp i\alpha\cdot\eta \mid \eta \in \Delta\}\,.$$

The choice of α also induces a reduction from SO(7) to SO(6) which is covered by $\mathrm{Spin}(6) = \mathrm{SU}(4)$, and as an SU(4)-module we have $\Delta^{1,0} = \mathbf{4}$ and $\Delta^{0,1} = \overline{\mathbf{4}}$. We define

$$\psi_\pm = \eta_\pm - i\alpha\cdot\eta_\pm$$

and let $\widetilde{g} = g_{|\widetilde{T}}$ and $\widetilde{b} = \alpha \lrcorner (\alpha\wedge b)$. Then a generalised SU(3)-structure over $\widetilde{T}$ is given by $(\widetilde{T}, \widetilde{g}, \widetilde{b}, \psi_\pm, \phi)$. Moreover, we get a (possibly zero) 1-form $\beta = \alpha \lrcorner b \in \Lambda^1\widetilde{T}^*$. It is clear that we can reverse this construction by defining a metric $g = \widetilde{g} + \alpha\otimes\alpha$, $b = \widetilde{b} + \alpha\wedge\beta$ and two spinors $\eta_\pm \in \mathbf{8}$ through $\eta_\pm = \mathrm{Re}(\psi_\pm)$.

To see what happens in the form picture, we start with the special $G_2\times G_2$-invariant form

$$\rho_0 = (\eta_+\otimes\eta_-)^{\mathrm{ev}} = f_0 + \alpha\wedge f_1,$$

where $f_0 \in \Lambda^{ev}\widetilde{T}^*$ and $f_1 \in \Lambda^{od}\widetilde{T}^*$. It follows from (4.2) that

$$\begin{aligned}\alpha\wedge(\eta_+\otimes\eta_-)^{\mathrm{ev,od}} &= \frac{1}{2}(\alpha\cdot\eta_+\otimes\eta_- \mp \eta_+\otimes\alpha\cdot\eta_-)^{\mathrm{od,ev}}\\ \alpha\lrcorner(\eta_+\otimes\eta_-)^{\mathrm{ev,od}} &= \frac{1}{2}(-\alpha\cdot\eta_+\otimes\eta_- \mp \eta_+\otimes\alpha\cdot\eta_-)^{\mathrm{od,ev}}\,.\end{aligned}$$

Therefore the forms f_0 and f_1 can be expressed by

$$f_0 = \alpha \lrcorner (\alpha \wedge \rho_0) = \frac{1}{2}(\eta_+ \otimes \eta_- + \alpha \cdot \eta_+ \otimes \alpha \cdot \eta_-)^{\mathrm{ev}}$$

and

$$f_1 = \alpha \lrcorner \rho_0 = -\frac{1}{2}(\alpha \cdot \eta_+ \otimes \eta_- + \eta_+ \otimes \alpha \cdot \eta_-)^{\mathrm{od}} .$$

Using the spinors $\psi_\pm$ as defined above we find

$$\begin{aligned} \psi_+ \otimes \psi_- &= (\eta_+ \otimes \eta_- - \alpha \cdot \eta_+ \otimes \alpha \cdot \eta_-) - i(\alpha \cdot \eta_+ \otimes \eta_- + \eta_+ \otimes \alpha \cdot \eta_-) \\ \psi_+ \otimes \overline{\psi}_- &= (\eta_+ \otimes \eta_- + \alpha \cdot \eta_+ \otimes \alpha \cdot \eta_-) + i(-\alpha \cdot \eta_+ \otimes \eta_- + \eta_+ \otimes \alpha \cdot \eta_-) . \end{aligned}$$

Consequently, we have

$$f_0 = \frac{1}{2}\,\mathrm{Re}(\psi_+ \otimes \overline{\psi}_-) = \tau_0$$

and

$$f_1 = \frac{1}{2}\,\mathrm{Im}(\psi_+ \otimes \psi_-) = \widehat{\upsilon}_0 .$$

In the same vein, decomposing $\square_{g,0}\rho_0 = g_1 + \alpha \wedge g_0$ yields

$$g_0 = \frac{1}{2}\,\mathrm{Im}(\psi_+ \otimes \overline{\psi}_-) = \widehat{\tau}_0$$

and

$$g_1 = \frac{1}{2}\,\mathrm{Re}(\psi_+ \otimes \psi_-) = -\upsilon_0 .$$

In presence of a non-trivial B-field $b \in \Lambda^2 T^*$ we write $b = \widetilde{b} + \alpha \wedge \beta$. Since $e^{\widetilde{b} + \alpha\wedge\beta} = e^{\widetilde{b}} \wedge (1 + \alpha \wedge \beta)$ we obtain for the general case the expressions

$$\rho_b = e^{-\phi}\tau_{\widetilde{b}} + \alpha \wedge \left(e^{-\phi}\widehat{\upsilon}_{\widetilde{b}} + \beta \wedge e^{-\phi}\tau_{\widetilde{b}}\right) \tag{7.1}$$

and

$$\square_{g,b}\rho = -e^{-\phi}\upsilon_{\widetilde{b}} + \alpha \wedge \left(e^{-\phi}\widehat{\tau}_{\widetilde{b}} - \beta \wedge e^{-\phi}\upsilon_{\widetilde{b}}\right) . \tag{7.2}$$

Conversely, if (T, g, b, ρ_0, ϕ) defines a generalised G_2-structure and $\alpha \in T$ is a unit vector, then the forms $\widetilde{b} = \alpha \lrcorner b$, $\tau_0 = \alpha \lrcorner (\alpha \wedge \rho_0)$ and $\upsilon_0 = -\alpha \lrcorner \square_{g,0}\rho_0$ define a generalised SU(3)-structure $(\widetilde{T}, \widetilde{g}, \widetilde{b}, \tau_0, \upsilon_0, \phi)$ with $\widetilde{g} = g_{|T}$.

To see how the integrability conditions relate to each other over the manifolds $M^7 = M$ and $M^6 = \widetilde{M}$, consider a smooth family $(\widetilde{g}(t), \widetilde{b}(t), \tau_0(t), \upsilon_0(t), \phi(t))$ of metrics $\widetilde{g}(t)$, of 2-forms $\widetilde{b}(t)$, of even and odd forms $\tau_0(t)$ and $\upsilon_0(t)$ and of scalar functions $\phi(t)$ which we assume to define a generalised SU(3)-structure for any t lying in some open interval I. Moreover, we consider a curve of 1-forms $\beta(t) \in \Omega^1(\widetilde{M})$. In order to obtain an integrable generalised G_2-structure over $\widetilde{M} \times I$ defined by $(\widetilde{M} \times I, g, b, \rho_0, \phi)$ where $g = \widetilde{g}_t \oplus dt \otimes dt$, $b = \widetilde{b}(t) + dt \wedge \beta(t)$, $\rho_0 = \tau_0(t) + dt \wedge \widehat{\upsilon}_0(t)$ and $\phi = \phi(t)$, we need to solve the equations

$$d\rho = 0 , \qquad d\square_{g,b}\rho = 0 .$$

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We decompose the exterior differential d over $M = \widetilde{M} \times I$ into

$$d_{|\Omega^{\rm ev,od}} \cdot \to d_{|\Omega^{\rm ev,od}} \cdot = \widetilde{d}_{|\Omega^{ev,od}} \cdot \pm \partial_{t|\Omega^{\rm ev,od}} \cdot \wedge dt\,,$$

where $\widetilde{d}$ is the exterior differential on $\widetilde{M}$. From (7.1) we conclude the first equation to be equivalent to

$$d\rho = \widetilde{d}e^{-\phi}\tau_{\widetilde{b}} + dt \wedge \left(\partial_t e^{-\phi}\tau_{\widetilde{b}} - \widetilde{d}e^{-\phi}\widehat{\upsilon}_{\widetilde{b}} - \widetilde{d}(\beta \wedge e^{-\phi}\tau_{\widetilde{b}})\right)$$

so that

$$\widetilde{d}e^{-\phi}\tau_{\widetilde{b}} = 0\,, \qquad \partial_t e^{-\phi}\tau_{\widetilde{b}} = \widetilde{d}e^{-\phi}\widehat{\upsilon}_{\widetilde{b}} + \widetilde{d}\beta \wedge e^{-\phi}\tau_{\widetilde{b}}\,. \tag{7.3}$$

By (7.2) the second equation reads

$$d\Box\rho = -\widetilde{d}e^{-\phi}\tau_{\widetilde{b}} + dt \wedge \left(-\partial_t e^{-\phi}\upsilon_{\widetilde{b}} - \widetilde{d}e^{-\phi}\widehat{\tau}_{\widetilde{b}} - \widetilde{d}(\beta \wedge e^{-\phi}\upsilon_{\widetilde{b}})\right)$$

and therefore yields

$$\widetilde{d}e^{-\phi}\upsilon_{\widetilde{b}} = 0\,, \qquad \partial_t e^{-\phi}\upsilon_{\widetilde{b}} = -\widetilde{d}e^{-\phi}\widehat{\tau}_{\widetilde{b}} + \widetilde{d}\beta \wedge e^{-\phi}\upsilon_{\widetilde{b}}\,. \tag{7.4}$$

If we let $\bar{\beta}(t) = \int_0^s \beta(s)ds$ we can bring (7.3) and (7.4) into hamiltonian form, that is the generalised G_2-structure is integrable if and only if

$$\begin{aligned}
\widetilde{d}\left(e^{-\phi}e^{\widetilde{d}\bar{\beta}} \wedge \upsilon_{\widetilde{b}}\right) &= 0\,, \qquad & \widetilde{d}\widehat{\upsilon} &= \partial_t\left(e^{-\phi}e^{\widetilde{d}\bar{\beta}} \wedge \tau_{\widetilde{b}}\right),\\
\widetilde{d}\left(e^{-\phi}e^{\widetilde{d}\bar{\beta}} \wedge \tau_{\widetilde{b}}\right) &= 0\,, \qquad & \widetilde{d}\widehat{\tau} &= -\partial_t\left(e^{-\phi}e^{\widetilde{d}\bar{\beta}} \wedge \upsilon_{\widetilde{b}}\right).
\end{aligned} \tag{7.5}$$

We illustrate the previous discussion by considering a classical SU(3)-structure defined by a unit spinor η and taking the generalised SU(3)-structure given by (M^6, g, η) with trivial B-field and vanishing dilaton, i.e. $b = 0$ and $\phi = 0$. We can compute the forms τ_0 etc. by using the normal form description (6.2) and (6.3) where $c_1 = 1$ and $c_2 = 0$. Using the notation of [2] and section 4 we obtain

$$\begin{aligned}
\upsilon_0 &= -\frac{1}{2}\psi_-\,, \qquad & \widehat{\upsilon}_0 &= \frac{1}{2}\psi_+\,,\\
\tau_0 &= \frac{1}{2}\left(1 - \frac{\omega^2}{2}\right), \qquad & \widehat{\tau}_0 &= \frac{1}{2}\left(\omega - \frac{\omega^3}{6}\right),
\end{aligned}$$

and equations (7.5) become the Hitchin flow equations

$$\begin{aligned}
\widetilde{d}\psi_- &= 0\,, \qquad & \widetilde{d}\psi_+ &= -\partial_t\omega \wedge \omega\,,\\
\widetilde{d}\omega \wedge \omega &= 0\,, \qquad & \widetilde{d}\omega &= \ \ \partial_t\psi_-\,,
\end{aligned}$$

which appeared in [2] and go back to [13]. Note that although equations (7.5) are, like the Hitchin flow equations, in hamiltonian form we have not shown yet that if the data $(\widetilde{g}(t), \widetilde{b}(t), \tau_0(t), \upsilon_0(t), \phi(t))$ defines a generalised SU(3)-structure at $t = t_0$ and satisfies (7.5), then it automatically defines a generalised SU(3)-structure for $t > t_0$, as it is the case for classical SU(3)-structures evolving along the Hitchin flow.

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